# A SIMPLE DISCUSSION OF LOGARITHMIC ERRORS.

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The errors of logarithmic computations arise primarily from the fact that the tabular mantissas are not accurate values of the functions: though their errors are governed by laws, we may assume that if an indefinitely large number of tabular mantissas be taken, distributed uniformly throughout the table, the average of their errors, disregarding the signs, will be a 0.25 of a unit of the last decimal place. Assuming that one error is as likely to occur as another, we find that there will be as many errors above 0.25 as below it. In the discussion to follow we shall employ the following law:—

*The probable error of the sum of any number of quantities equals the square root of the sum of the squares of the probable errors of those quantities.*

## PROBLEM I.

*To find the probable error of a mantissa taken from a 4, 5, or 6-place table of logarithms of natural numbers.*

In the best 4, 5, or 6-place tables the printed argument contains one less figure than the corresponding function. But in computation we have, in general, as many figures in the natural numbers as in the mantissas of their logarithms. Hence in interpolation, the tabular difference is multiplied by a decimal which ranges from 0.1 to 0.9. Represent this decimal by $x$: let two consecutive values of tabular mantissas be $a$ and $b$, $a + c$ and $b + c'$ being their true values: the apparent tabular difference is then $b - a$, its true value being $b - a + c' - c$. The interpolated mantissa (if no figures of the result are rejected) is $a + x(b - a)$. It should be $a + c + x(b - a) + x(c' - c)$.

In using an $n$-place table, we reject all figures beyond the $n^{th}$, thus introducing a probable error of 0.25 of a unit in the $n^{th}$ place.

Therefore, the probable error of an interpolated mantissa, when carried to $n$ places is (since $c$ and $c'$ have a probable error of 0.25)

$$[(0.25)^2 + (0.25\,x)^2 + (0.25\,(1 - x))^2]^{\frac{1}{2}} = 0.25\,(2 - 2x + 2x^2)^{\frac{1}{2}}$$

We obtain thus a table of probable errors for different values of $x$.

| $x$ | Probable Error. |
|---|---|
| 0.1 | $\sqrt{1.82} \times 0.25 = 0.337$ |
| 0.2 | $\sqrt{1.68} \times 0.25 = 0.324$ |
| 0.3 | $\sqrt{1.58} \times 0.25 = 0.314$ |
| 0.4 | $\sqrt{1.52} \times 0.25 = 0.308$ |
| 0.5 | $\sqrt{1.50} \times 0.25 = 0.306$ |

| | |
|---|---|
| 0.6 | $\sqrt{1.52} \times 0.25 = 0.308$ |
| 0.7 | $\sqrt{1.58} \times 0.25 = 0.314$ |
| 0.8 | $\sqrt{1.68} \times 0.25 = 0.324$ |
| 0.9 | $\sqrt{1.82} \times 0.25 = 0.337$ |

But $x$ is as likely to be 0.0 as one of the values of the preceeding table: in this case since no interpolation is performed, the probable error is 0.25. The average of these probable errors is 0.31.

I have taken six groups of ninety 4-place numbers each, and found the mantissas of their logarithms from a 4-place logarithm table. The errors of these mantissas were found by comparison with a 5-place table. The numbers were equally distributed throughout the table: care was taken to have for the final figures as many zeros as ones, twos, threes, etc. Four similar groups of ninety 5-place numbers each and one hundred numbers each of which contained six digits, were treated likewise. The average of the numerical values of the errors of the mantissas of each set was taken, and the results are given below, in units of the $n^{th}$ place, $n$-place logarithms.

| Sets. | Averages. |
|---|---|
| 90 4-place numbers. | 0.306 |
| " " | 0.302 |
| " " | 0.300 |
| " " | 0.301 |
| " " | 0.301 |
| " " | 0.291 |
| 90 5-place numbers. | 0.309 |
| " " | 0.309 |
| " " | 0.327 |
| " " | 0.297 |
| " " | 0.292 |
| 100 6-place numbers. | 0.299 |

The number of errors whose values were 0.0, 0.1, etc., are exhibited below:

| Errors. | No. of Errors. | Errors. | No. of Errors. |
|---|---|---|---|
| 0.0 | 97 | 0.5 | 93 |
| 0.1 | 197 | 0.6 | 53 |
| 0.2 | 171 | 0.7 | 46 |
| 0.3 | 156 | 0.8 | 28 |
| 0.4 | 149 | 0.9 | 10 |

The average of the perceding 1000 errors is

$$0.30$$

which is close to the probable error given by the theory.

PROBLEM II.

*Given a 4, 5, or 6-place mantissa, to find the probable error of the value obtained for the corresponding natural number.*

Let $l$ be the given mantissa, $t$ and $t'$ the tabular values of the next less, and next greater mantissas, $t + c$ and $t' + c'$ being their respective real values. Then $t' - t$ is the apparent tabular difference, and $t' - t + c' - c$ the true one. To get the last figure of the natural number corresponding to the mantissa $l$, we divide $l - t$ by $t' - t$. Accuracy would be reached by dividing $l - (t + c)$ by $t' - t + c' - c$. Denote $t' - t$ by $D$, and $l - t$ by $d$.

$$\frac{d - c}{D + (c' - c)} = \frac{d}{D} - \frac{d}{D^2}c' - \frac{1 - \frac{d}{D}}{D}c + \frac{1 - \frac{d}{D}}{D^2}c(c' - c) + \frac{d}{D^3}c'(c' - c) + \ldots$$

In practice we employ the fraction $10\frac{d}{D}$ to get the last figure of the natural number, but since we reject all figures of the quotation, after the first, we may be said to use $10\frac{d}{D} \pm 0.25$, $0.25$ being the probable value of the rejected figures. Therefore our error is, with sufficient approximation

$$10\frac{d}{D^2}c' + 10\frac{1 - \frac{d}{D}}{D}c \pm 1.25.$$

The probable values of $c$ and $c'$ being $0.25$ we have for the probable value of this expression

$$\left[100\frac{\left(\frac{d}{D}c'\right)^2 + \left[\left(1 - \frac{d}{D}\right)c\right]^2}{D^2} + (0.25)^2\right]^{\frac{1}{2}}$$

$$= \left(\left[\frac{10}{D}\sqrt{\left(\frac{d}{D}c'\right)^2 + \left[\left(1 - \frac{d}{D}\right)c\right]^2}\right]^2 + (0.25)^2\right)^{\frac{1}{2}}$$

$$= \left(\left[\frac{2.5}{D}\sqrt{\left(\frac{d}{D}\right)^2 + \left(1 - \frac{d}{D}\right)^2}\right]^2 + (0.25)^2\right)^{\frac{1}{2}}$$

Now $\frac{d}{D}$ ranges from 0.1 to 0.9. When $\frac{d}{D} = 0.1$, 0.2, etc. we get for $\sqrt{\frac{d^2}{D^2} + \left(1 - \frac{d}{D}\right)^2}$ the successive results 0.906, 0.825, 0.762, 0.721, 0.707, 0.721, 0.762, 0.825, and 0.906, whose average is 0.793. Hence, with sufficient accuracy, the formula for the average of the probable errors becomes

$$\left[\left(\frac{1.98}{D}\right)^2 + (0.25)^2\right]^{\frac{1}{2}}$$

If one is computing in any part of the table, he can obtain the average probable error quite closely by substituting for $D$ the arithmetical mean of the tabular differences employed. If the *numbers* corresponding to the given mantissas be uniformly distributed throughout the table, the average of all the values of the term $\frac{1.98}{D}$ will be $\frac{1.98}{8}$ or 0.25. Substituting this in the formula we get 0.35 for the average probable error. But if the mantissas be uniformly distributed, the average value of $\frac{1.98}{D}$ is 0.18; this substituted in the formula gives 0.31.

The foregoing formula takes no account of the fact that the given mantissa is often found printed in the table. In that case the probable error would be $\frac{2.5}{D}$ in units of the $n^{th}$ place.

I have taken a set of ninety uniformly distributed 4-place mantissas and found the corresponding four-place numbers. The average of the errors was 0.28. It happened that fifteen of the chosen mantissas were given in the table a much larger proportion than would be expected. These tended to diminish the average error.

### PROBLEM III.

*To adapt Problems* I *and* II *to a 7-place logarithm table, or a table in which the argument has two less figures than the function.*

The discussion of Problem I would be changed by using 0.01, 0.02, . . . 0.99 for the values of $x$, and in Problem II these would be substituted for $\frac{d}{D}$. It is evident that the numerical results obtained would not be materially altered.

The reasoning of the two preceding problems will apply to any tables in which the successive tabular differences are so nearly equal that second differences may be neglected.

### PROBLEM IV.

*Given an n-place natural number: to find the maximum error of an n-place interpolated mantissa.*

The expression $xc' + (1 - x)c$ was found in Problem I, and this is the actual error of a mantissa interpolated from an n-place table if the mantissa be carried beyond the $n^{th}$ place. If it be cut off at the $n^{th}$ place an error which may amount to 0.5 of a unit of the $n^{th}$ place may be committed. That the expression may be a maximum, both $c$ and $c'$ must be maxima and must have the same sign. Making the signs the same throughout, we have

$$\begin{aligned}\text{Maximum error} &= 0.5\,x + 0.5\,(1 - x) + 0.5 \\ &= 1.0.\end{aligned}$$

This error was not reached in any one of the thousand examples mentioned under Problem I.

PROBLEM V.

*Given an n-place mantissa to find the maximum error of the corresponding natural number.*

Neglecting the terms which involve $c' - c$ in the first equation of Problem II, multiplying the terms by $10^f$, and supposing that the natural number is carried beyond the $n^{th}$ place, we have

$$\text{Error} = \left[\frac{d}{D^2}c' + \frac{1 - \frac{d}{D}}{D}c\right]10^j,$$

where $f$ is the difference between the number of figures in the argument and those in the function. To have a maximum error for any given values of $d$ and $D$, $c$ and $c'$ must be maxima and have the same sign. Then,

$$\text{Maximum error} = \frac{0.50}{D}\,10^f.$$

When $f$ is unity, as in the ordinary 4, 5, and 6-place tables, we have the error equal to $\frac{5}{D}$. In these tables the smallest value of $D$ compatible with the condition that $c = c'$ is 5. But since an error of 0.5 may be introduced, when all figures beyond tne $n^{th}$ are rejected, we have in this case

$$\text{Maximum error} = 1.50.$$

In an ordinary 7-place table $f = 2$ and the smallest value of $D$ compatible with the condition that $c = c'$ is 44. Hence in this case the maximum error is

$$\frac{50}{44} + 0.5 = 1.64.$$